When not to use l'Hopital's rule

l'Hopital's rule is very popular because it promises an automatic way of computing limits of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \text{“}\frac{0}{0}\text{”}.$$

Instead of figuring out a trick that transforms the fraction into some other expression whose limit we can compute, l'Hopital tells us just to differentiate both $f(x)$ and $g(x)$ and then try again (repeat as necessary). There are many examples where l'Hopital's rule is the simplest way to an answer. But there are also many examples where l'Hopital makes you work much harder than necessary, and even a few where l'Hopital just doesn't get you to the answer.

1. Example – Compute $\lim_{x\to 0} \frac{\sin x^3}{\sin^3 x}$

The limit

$$\lim_{x\to 0} \frac{\sin x^3}{(\sin x)^3} \tag{1}$$

is of the form $\frac{0}{0}$ so l'Hopital's rule applies. Recall that this rule says that if

$$\lim_{x\to 0} f(x) = 0 \text{ and } \lim_{x\to 0} g(x) = 0$$

then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)},$$

provided the second limit exists.

Let's see how useful l'Hopital is in computing $\lim_{x\to 0}(\sin x^3)/(\sin x)^3$.

Without l'Hopital's rule.

$$\lim_{x\to 0} \frac{\sin x^3}{\sin^3 x} = \lim_{x\to 0} \frac{\sin x^3}{x^3}\,\frac{x^3}{\sin^3 x} = \lim_{x\to 0} \frac{\sin x^3}{x^3}\left(\frac{x}{\sin x}\right)^3 = 1\cdot 1^3 = 1. \qquad \textit{Done!} \tag{2}$$

With l'Hopital's rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x^3}{\sin^3 x} &= \text{“}\frac{0}{0}\text{”} && \textit{so we can use l'Hopital}\\
&= \lim_{x\to 0} \frac{3x^2\ \cos x^3}{3\sin^2 x\ \cos x} && \textit{differentiated top \& bottom}\\
&= \text{“}\frac{0}{0}\text{”} && \textit{so we can use l'Hopital again}\\
&= \lim_{x\to 0} \frac{6x\cos x^3 - 9x^4 \sin x^3}{6\sin x\ \cos^2 x - 3\sin^3 x} && \textit{differentiated top \& bottom}\\
&= \text{“}\frac{0}{0}\text{”} && \textit{so we can use l'Hopital again}\\
&= \lim_{x\to 0} \frac{6\cos x^3 - 18x^3\sin x^3 - 36x^3\sin x^3 - 27x^6\cos x^3}{6\cos^3 x - 12\sin^2 x\ \cos x - 9\sin^2 x\ \cos x} && \textit{differentiated top \& bottom}\\
&= \frac{6}{6}\\
&= 1. && \textit{Done!}
\end{aligned}$$

2. Other examples

Here are two limits that are all of the form $\frac{0}{0}$, and to which one could in principle apply l'Hopital's rule. For these particular limits l'Hopital leads to severe headaches, while there are simple ways of finding the limits.

$$I = \lim_{x\to 0} \frac{\sin x^5}{\sin^5 x}$$

$$J = \lim_{x\to\infty} \frac{e^x}{e^x + e^{-x}}$$

Both of these limits is equal to 1.

3. Solutions

The limit I is just like the one we did when we computed (1). Using l'Hopital you get a really long computation, but you can also compute it as in (2).

The limit J just turns into itself after applying l'Hopital twice. The better way is to divide top and bottom by e^x (the biggest term around).