Statement of Purpose

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Arriving at the Badger University, I had seen a decent amount of applied math; my courses in high school covered three semesters of calculus and one each of differential equations and linear algebra. However, as I quickly learned, I had not even scratched the surface of proof-based math. Fortunately, this began to be rectified during my first year Honors Calculus sequence, which through three quarters provided a rigorous introduction to basic mathematical analysis. My interest in analysis was solidified and even enhanced by the sequence I took during my second year: honors analysis.

Honors analysis has a reputation as one of the most demanding undergraduate courses at the Badger University, and the courses more than lived up to that reputation. During the first quarter, I found myself working between twenty and thirty hours a week on that course alone, every week. We covered almost the entirety of Rudin's *Principles of Mathematical Analysis*, linear algebra and somewhat less introductory topics like basic spectral theory and functional analysis in just the first quarter. The second quarter focused mainly on differentiation and integration in $\mathbb{R}^n$, but also extended to integration on chains and manifolds, as well as a look at smooth manifolds. However, the most significant quarter for me was the third of the sequence.

Measure theory, the Lebesgue integral, and L^p spaces were the main topics during this third quarter, but there was also significant time to cover more specific concepts and results that the professor, Florence Bascom, selected. We proved the Banach-Tarski paradox at start of the class, studied applications of the Baire category theorem and various covering lemmas, examined Banach-Mazur games, and then closed the course by looking at Hausdorff dimension. A common theme was the pervasiveness of pathological examples. The construction of Vitali sets shows that the collection of measurable subsets is much smaller than the power set, and Weierstrass's example is more typical of continuous functions than differentiability almost everywhere. Additionally, Professor Bascom provided challenging optional problems at various points during the lectures, including one that she did not inform us at the time was open. I was the only one in the class to complete one of these (sadly not the open one), by constructing a continuous function for which every level set was a Cantor set.

I am a physics major as well as a math major. Entering college, I loved both fields, and I had no way of determining which I would want to pursue academically except by delving into both and being patient. As I saw more and more of each, I gravitated towards math and soon could not imagine myself doing anything else. Honors analysis 3 was the most important step towards this. The work was not just tolerable or even pleasant, but actively exciting. One of the main reasons I set out to take the honors analysis sequence was that I needed to test myself to see how I would feel about math at a higher intensity level. Would it exhaust or invigorate me? If I wanted to work in math academically, I needed to know.

By the end of my second year, the answer was quite clear. That summer, I participated in the math REU at the Badger University, and expanded my interests by studying ergodic theory and some dynamics. I registered for the graduate analysis sequence, as well as the honors algebra sequence to take during my third year. Algebra was quite interesting, but my passion remained most prominent for analysis. The graduate sequence covered probability in the first quarter, and then functional analysis in the second, the latter of which I particularly enjoyed. Before the third quarter, I had a conversation with Professor Bascom, and discovered that she would be having a course on geometric measure theory with some of her graduate students and postdocs, which I was thankfully allowed to register for. The format was new for me and a bit daunting, consisting of each student delivering three eighty-minute lectures. We were all instructed to select three topics that we did not know much about, but as a complete novice in GMT at the time, that was not an issue for me.

The first presentation I gave in this course was on Cantor measures and Pisot numbers. It was fascinating to learn how examining a type of measure could naturally give rise to a concept usually defined in algebraic terms. However, the second presentation on Besicovitch sets piqued my interest even more significantly. Starting the course, I was completely unfamiliar with the concept, and would probably have naively guessed that the construction of a null set with a line segment in every direction would be impossible. Not only is it possible, but the existence of Besicovitch sets has been applied to produce seemingly unconnected results, notably Fefferman's famous proof that the disc multiplier operator is not L^p bounded in the plane for $p \neq 2$. The third presentation focused on constructing fractional Sobolev spaces using techniques from Fourier analysis, which for me was significantly more intuitive than the construction we had used the previous quarter in graduate analysis.

At least as instructive as giving the lectures, however, was listening to the others. As the only undergraduate, I was the least experienced with the material, but I also had less experience organizing my talks, presenting effectively, and addressing questions. I was able to learn from people who were almost exactly where I wanted to be in two, four, or six years. Their specific topics were extremely engaging, and indicated so many avenues for future study: Riesz products, Bernoulli convolutions, oscillatory integrals amongst many others. At the end of the course, Professor Bascom served as my mentor for the summer's REU, and along with another student, we studied different constructions and properties of Besicovitch sets.

With this topic, it felt like there were many possible directions to proceed. Of course, there is the intractable Kakeya conjecture, but considering $\mathbb{R}^n$ instead of $\mathbb{R}^2$ is not the only logical generalization. In particular, one can replace unit line segments with rectifiable curves of length one, and stipulate that copies of these curves rotated by every angle appear in some subset of the plane to define a sort of generalized Besicovitch set. Due to the recent work of Chang and Csörnyei, it is known that such sets of measure zero exist in the plane for every rectifiable unit length curve, but results pertaining to the Hausdorff dimension of these sets are less thorough [1]. Taking the curve to be the graph of the standard Cantor function restricted to the Cantor set, for instance, nothing better than the trivial lower bound of dimension one is known for the resultant generalized Besicovitch sets. There are numerous closely related problems, including whether extending a set of line segments to a set of lines can ever increase the Hausdorff dimension of the set [2].

In addition to research, I am also motivated to pursue math for the teaching opportunities it offers. I have worked as a math tutor for the last five years. I enjoy educational outreach opportunities more broadly, and my most significant commitment aside from math is serving as the Secretary-General of the Model United Nations of the Badger University. Each year, we host almost three thousand delegates from around the world in Buckyland, as well as run numerous international programs. Rather than focus on the competitive aspects of Model UN, the organization takes a pedagogical approach and seeks to help delegates develop their public speaking, debate, and policy skills. The proudest part of our entire organization for me is the outreach to local high schools we perform, each week sending our members to teach small groups of students in the Hyde Park area, which I have personally done for the last three years. I would love to do the same for math, and if admitted I would be eager to participate in any departmental outreach.

Throughout my experience with math thus far, the most interesting ideas are those that challenge naive intuition. Nobody would guess the full intricacies of any field of math from the outset, and counterintuitive results are commonplace as well as instructive. It is fascinating to unfold an idea and explore all the consequence, foreseeable and unforeseeable. Although at the start of my time in college I was uncertain, the past few years have left no doubt that I want to study and research mathematics for the rest of my life, all while developing productive relationships with mentors, colleagues, and eventually students. UW Madison's mathematical reputation is undoubtedly eminent, and this is particularly true in analysis. Madison faculty members have a tremendous variety of specialities, and although I have mostly studied geometric measure theory thus far, I am eager to gain further experience with functional analysis, PDEs, and harmonic analysis, as well as to explore other related fields such as dynamical systems. I look forward to maturing as a mathematician in a community rigorously dedicated to research with motivated colleagues and skilled instructors, and such a community clearly exists at UW Madison.

References

[1] Alan Chang, Marianna Csörnyei, *The Kakeya needle problem and the existence of Besicovitch and Nikodym sets for rectifiable sets*, Proc. London Math. Soc. 118 (2019), 1084-1114.

[2] Tamás Keleti, *Are lines much bigger than line segments?*, Proc. Amer. Math. Soc. 144 (2016), 1535-1541.